Modeling the Sonority of Chord Progressions: Toward a psychophysical explanation of the "rules" of traditional harmony theory

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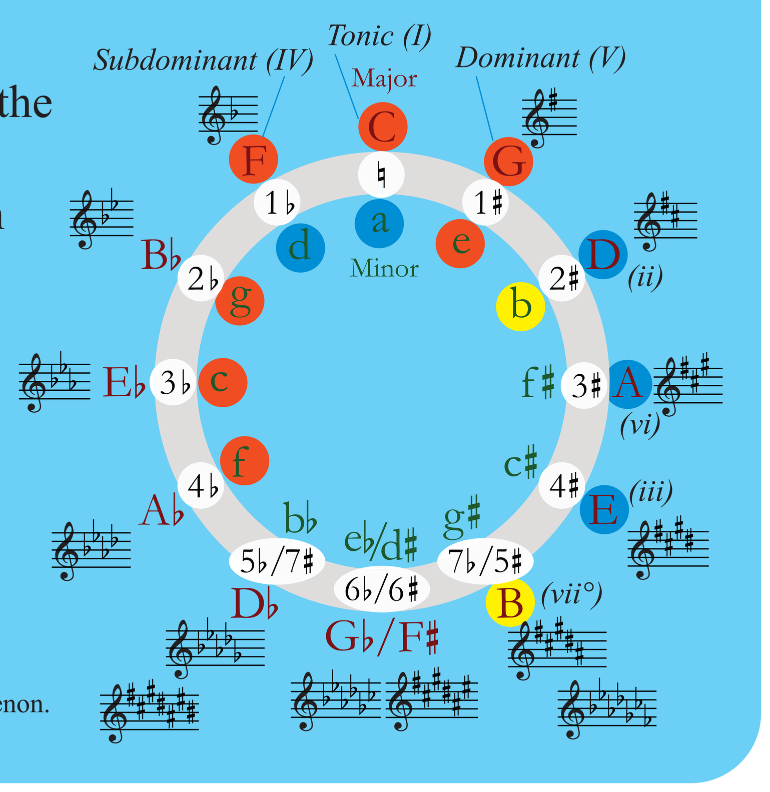
Motivation

The purpose of the present study was to calculate the overall "sonority" of triad chord progressions, using a psychophysical model of harmony perception.[1] We seek to explain why certain chord progressions are favored in so-called "Western music." In particular, we intend to find a psychophysical justification for the pattern of key relationships known as the *Circle of Fifths*.

Our research shows that one of the most basic rules of traditional harmony theory, as expressed in the Circle of Fifths, is not an arbitrary social construct, but, on the contrary, has a firm foundation in the acoustical structure of the chords themselves.



[1] Cook, N.D. & Fujisawa, T.X. The psychophysics of harmony perception: Harmony is a three-tone phenomenon. *Empirical Musicology Review* **1**, 106-126 (2006)

Approach

We employ a computational approach based on mathematical models for dissonance and tension (defined below). All computations include 6 or 7 upper partials and were carried out in terms of semitone intervals.

Notations

We use the following notational conventions:

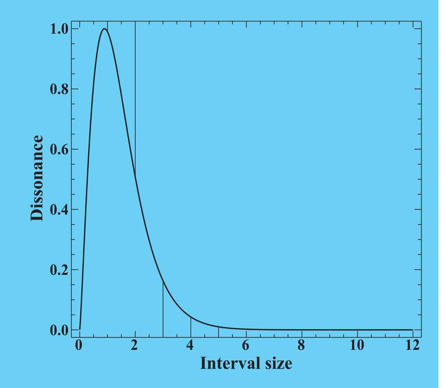
- We represent the i-th partial (overtone) of a tone with fundamental frequency f_1 as f_1^i;
- The interval x between two frequencies f_1 and f_2 is given by $x = 39.8631 \log(f_{\max}/f_{\min})$, where $f_{\max}$ is the larger of $[f_1, f_2]$ and $f_{\min}$ the smaller one;
- The timbre is parameterized as 0.88^i, where i labels the partial.

Dissonance

The ***dissonance***, d, between two frequencies is parameterized as:

$$d(f_1^i, f_2^j, i, j) = 0.88^{i+j} b_3 (e^{-b_1 x^\gamma} - e^{-b_2 x^\gamma}),$$

with $b_1 = 0.8$, $b_2 = 1.6$, $b_3 = 4.0$, and $\gamma = 1.25$.

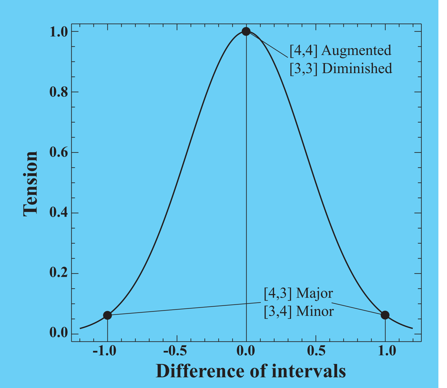


Tension

The ***tension***, t, of a triad chord is parameterized as a Gaussian curve:

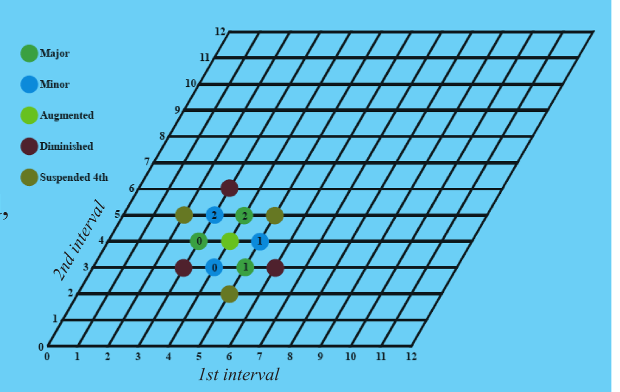
$$t(f_1, f_2, f_3, i, j, k) = 0.88^{i+j+k} e^{-\left(\frac{y-x}{\alpha}\right)^2},$$

with $\alpha = 0.6$; x is the interval between the lowest and middle frequencies, and y is the interval between the highest and middle frequencies; i, j, and k refer to the partials of each tone in the triad chord.



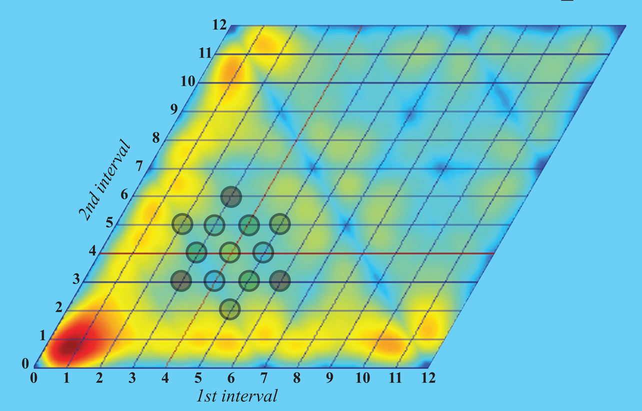
Hexagonal semitone coordinates

To display the computational results for dissonance and tension of triad chords, we will employ a hexagonal reference frame, with semitone divisions along each coordinate axis. The grid on the right shows this reference frame, along with the locations of major [4,3] and minor [3,4] chords (in root, 1st, and 2nd settings), the augmented [4,4] chord, and the diminished [3,3] and suspended 4th [5,5] chords, along with their different inversions.



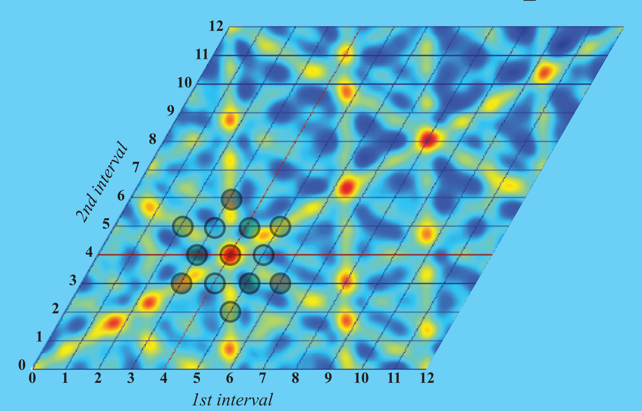
Results

Triad dissonance plot



All major, minor, diminished, suspended 4th, and the augmented chords are located in a region of relatively low dissonance.

Triad tension plot



All major and minor chords lie near local tension minima; augmented, diminished and suspended 4th chords lie on or near local tension maxima.

Two-Chord Progressions

We make the assumption that a similar mathematical approach, taking into account upper partials, will allow for the analysis of two-chord progressions. First we define the dissonance, tension, and sonority for a triad chord; then we extend the definition to chord progressions.

Triad Chord Dissonance

The total dissonance, D, for a pair of tones with fundamental frequencies f_1 and f_2, taking into account N upper partials, is defined as:

$$D(f_1, f_2; N) \equiv \sum_{i=0}^{N} \sum_{j=0}^{N} \left[\frac{1}{2} \left(d(f_1^i, f_1^j, i, j) + d(f_2^i, f_2^j, i, j) \right) + d(f_1^i, f_2^j, i, j) \right]$$

The total dissonance, D_c, for a triad chord can be computed simply by adding the dissonances of the pairs of frequencies:

$$D_c(f_1, f_2, f_3; N) = D(f_1, f_2; N) + D(f_1, f_3; N) + D(f_2, f_3; N)$$

Triad Chord Tension

The total tension, T, for a triad chord is computed as:

$$T(f_1, f_2, f_3; N) \equiv \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} t(f_1^i, f_2^j, f_3^k, i, j, k)$$

Triad Chord Instability

$$I(f_1, f_2, f_3; N) \equiv D_c(f_1, f_2, f_3; N) + 0.2T(f_1, f_2, f_3; N)$$

The sonority, S, of a triad chord is defined as the opposite (negative) of the instability.

Two-Chord Progression Instability/Sonority

For a sequence of two triad chords, the instability, I_2, is computed by taking the frequencies of the first (f) and the second (s) chord, and averaging the individual dissonances and tensions before combining them into the instability:

$$I_2(f, s; N) \equiv \frac{1}{9} \sum_{i=1}^{3} \sum_{j=1}^{3} D(f_{f,i}, f_{s,j}; N) + \frac{0.2}{18} \sum_{i=1}^{3} \sum_{j=1}^{2} \sum_{k=j+1}^{3} \left[T(f_{f,i}, f_{s,j}, f_{s,k}; N) + T(f_{s,i}, f_{f,j}, f_{f,k}; N) \right]$$

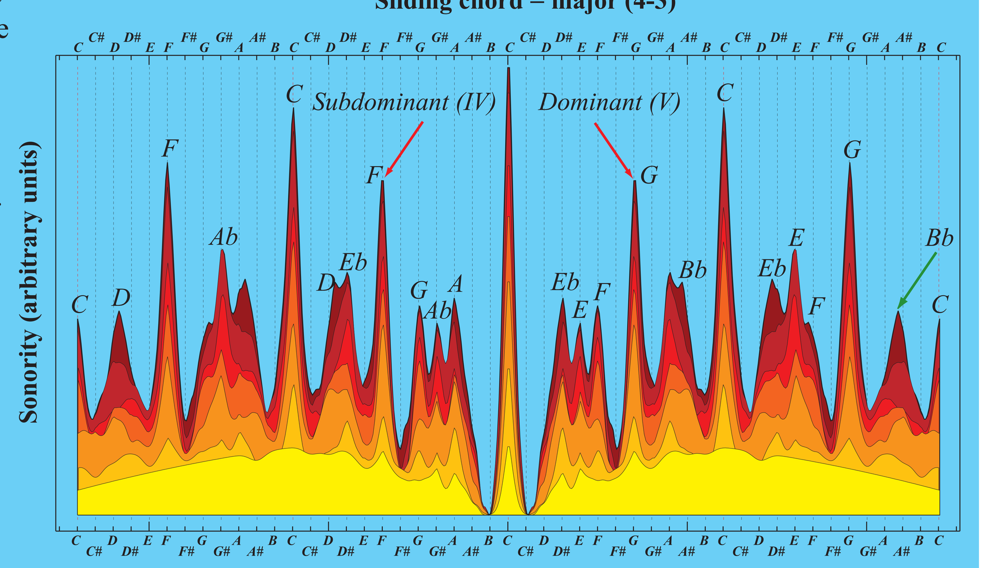
Results

We fix the first chord to be C-major (4-3); the second chord, also major, "slides" along the interval axis, starting with C-major two octaves below the fixed chord, and ending two octaves above. For each position of the second chord, the sonority is computed using a number of upper partials from zero (yellow) to six (dark brown).

Note that the sonority peaks at the subdominant (I-IV) and the dominant (I-V) chord progressions.

The other peaks correspond to the major chords that occur with C-major in various major and minor keys.

Fixed chord = C-major
Sliding chord = major (4-3)

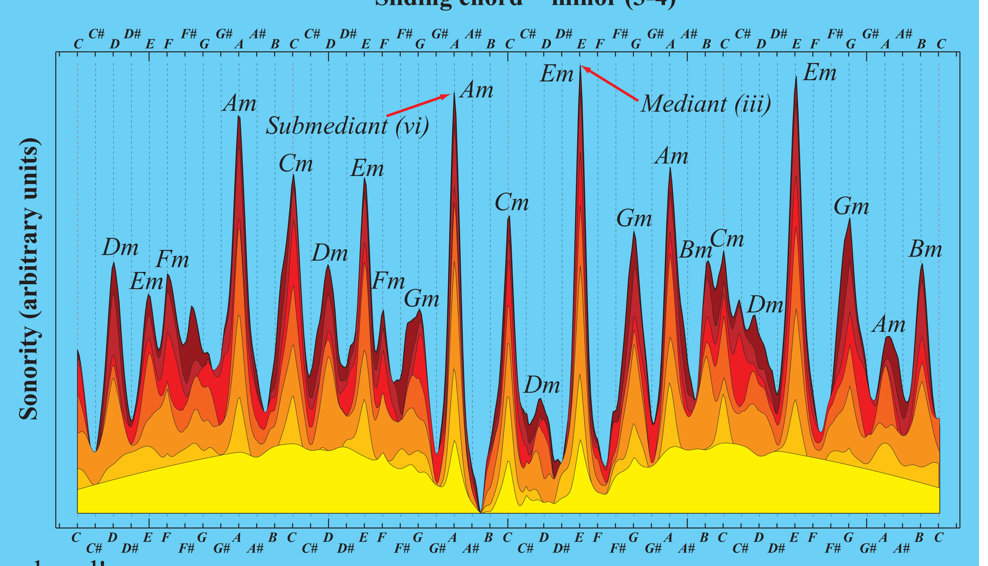


	M	m	m	M	M	m	dim
major	F	G	A	Bb	C	D	E
	C	D	E	F	G	A	B
	G	A	B	C	D	E	F#
	m	**dim**	**M**	**m**	**M**	**M**	**M**
minor	F	G	Ab	Bb	C	Db	Eb
	D	E	F	G	A	Bb	C
	A	B	C	D	E	F	G
	E	F#	G	A	B	C	D

All chords except B and Db/C# are reproduced!

When the sliding chord is a minor chord (3-4), then the sonority peaks at the mediant (I-iii) and submediant (I-vi) chord progressions.

Fixed chord = C-major
Sliding chord = minor (3-4)

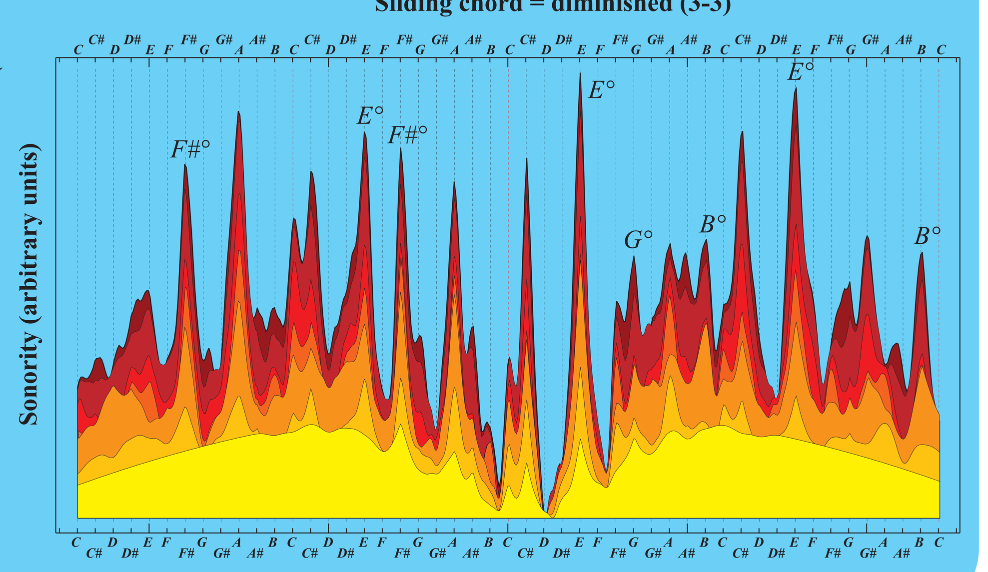


	M	m	m	M	M	m	dim
major	F	G	A	Bb	C	D	E
	C	D	E	F	G	A	B
	G	A	B	C	D	E	F#
	m	**dim**	**M**	**m**	**M**	**M**	**M**
minor	F	G	Ab	Bb	C	Db	Eb
	D	E	F	G	A	Bb	C
	A	B	C	D	E	F	G
	E	F#	G	A	B	C	D

All chords except Bbm are reproduced!

When the sliding chord is a diminished chord (3-3), then a variety of sonority peaks are found.

Fixed chord = C-major
Sliding chord = diminished (3-3)



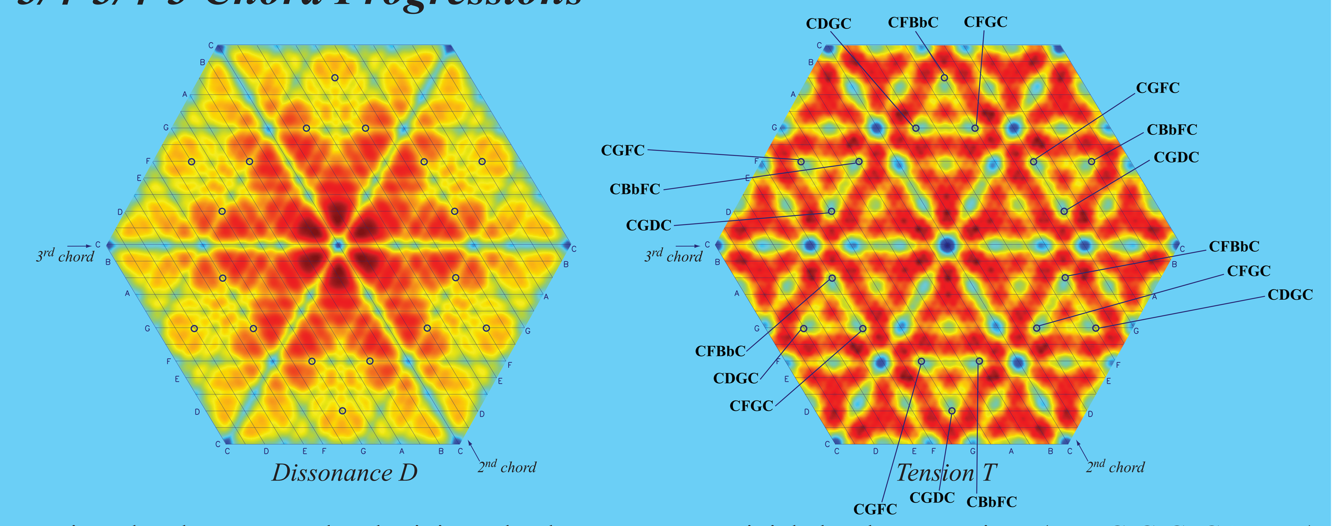
	M	m	m	M	M	m	dim
major	F	G	A	Bb	C	D	E
	C	D	E	F	G	A	B
	G	A	B	C	D	E	F#
	m	**dim**	**M**	**m**	**M**	**M**	**M**
minor	F	G	Ab	Bb	C	Db	Eb
	D	E	F	G	A	Bb	C
	A	B	C	D	E	F	G
	E	F#	G	A	B	C	D

All diminished chords plus a number of others are found!

Three-Chord Progressions

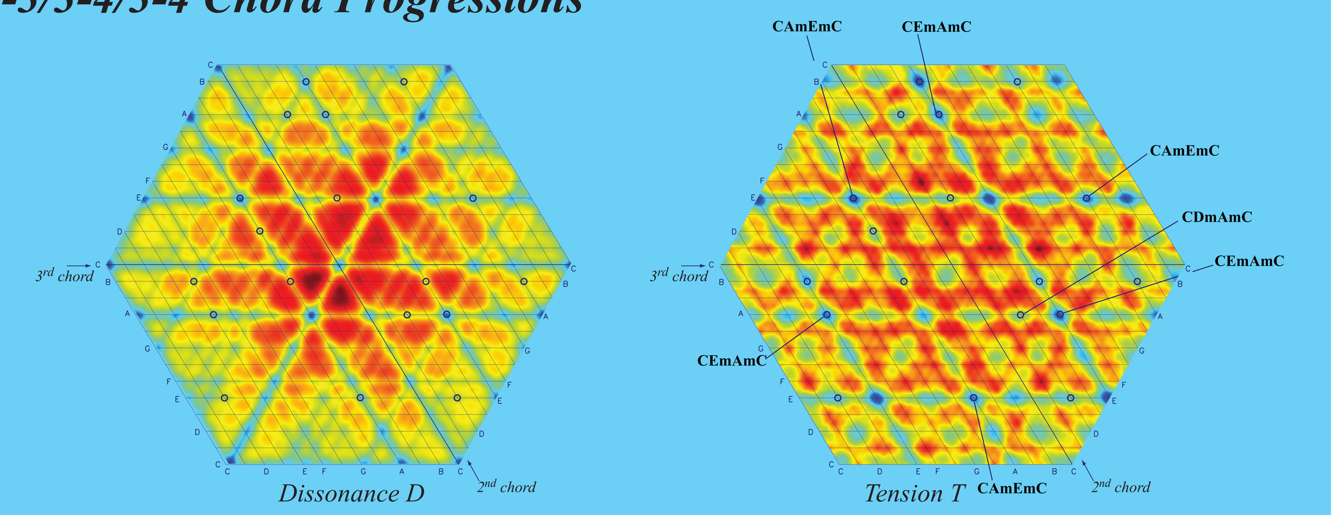
The two-chord progression approach can be extended to three chords, using the same mathematical relations. Since there are now three chords involved, with the first chord fixed to C-major, the resulting diagrams are two-dimensional and represented on a hexagonal grid.

4-3/4-3/4-3 Chord Progressions



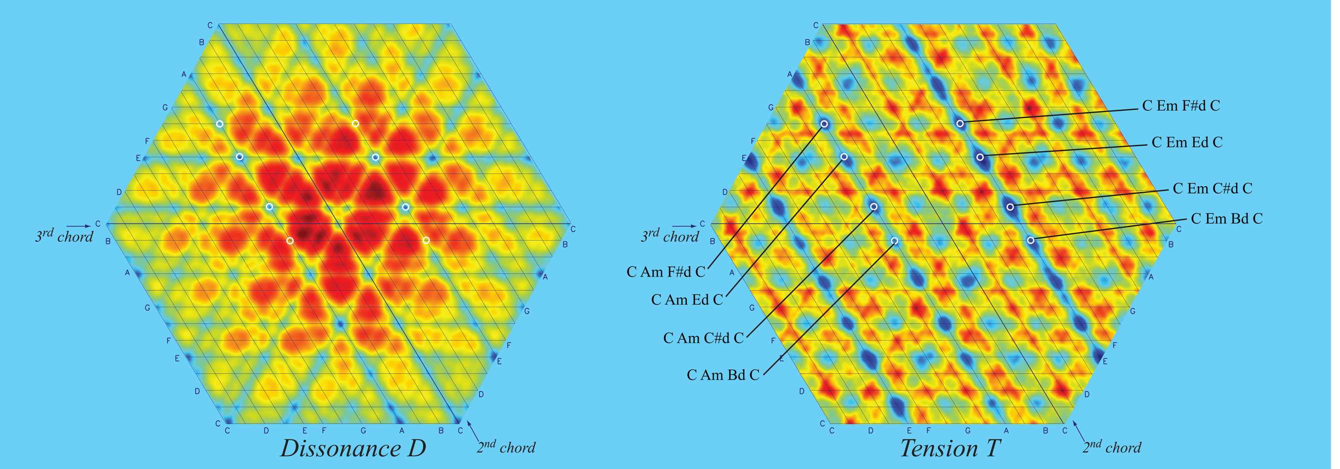
The tension plot shows many local minima, the deepest ones at trivial chord progressions (e.g., C-C-C, C-F-F, ...); in addition, there are triplets of minima, arranged in triangles, corresponding with the chord progressions of the major keys of F (F-Bb-C), C (C-F-G), and G (G-C-D). Most of these minima are also minima in the dissonance plot. There are many shallow minima, some of them corresponding to chord progressions from the minor keys, such as Dm (C-A-F), Am (C-E-G), Fm (C-Ab-Eb), and so on.

4-3/3-4/3-4 Chord Progressions



The tension plot shows local minima at all chord progressions from the major keys of F (C+[Gm,Am,Dm]), C (C+[Dm,Em,Am]), and G (C+[Am,Bm,Em]). In addition, several shallow minima correspond to progressions in the minor keys, e.g., Dm (C+[Dm,Gm]). The only minor key for which no minima are found for this particular chord progression is Fm (C+[Fm,Bbm]). As before, many, but not all, of the minima in the tension plot have corresponding minima in the dissonance plot.

4-3/3-4/3-3 Chord Progressions



The tension plot shows rows of local minima for chord progressions in which the second chord is Dm, Em, Gm, Am, and Bm. For the third chord, minima are found for Ed, Bd, and F#d, so that all chord progressions that are consistent with the major keys of F, C, and G, and the minor keys of Dm, Am, and Em are reproduced by the tension computation. The progressions in the key of Fm (C+[Fm,Bbm]+Gd) do not correspond to local minima.

Conclusions

Our research has shown that a computational approach to dissonance, tension, and sonority, which includes several upper partials in addition to the fundamental frequency, can reproduce several aspects of the traditional harmony theory (i.e., the Circle of Fifths), including the pattern of major and minor triad chords that belong to a given key, as well as "pleasant" three-chord chord progressions within a given key.

Many, but not all, of the minima in the tension plots for three-chord progressions coincide with minima in the dissonance plots, suggesting that tension (and, thus, sonority) may play an important role in identifying particularly sonorous chord progressions. This computational approach can be extended easily to arbitrary sequences of triad chords, as well as to chords with more than three tones. Our work supports the notion that harmony has an essentially acoustical basis.